

# SIMULTANEOUS VIBRATION AND FATIGUE OPTIMIZATION OF AN INERTER-BASED VIBRATION ISOLATION SYSTEM

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**Abstract:** This paper presents an analysis of vibration-induced fatigue in a generalized, two degree-of-freedom vibration isolation system. The system consists of a source body and a receiving body, coupled through a passive isolator. The isolator consists of a spring, a damper, and an inerter. A broad frequency band excitation of the source body is assumed. Optimized system, in which the kinetic energy of the receiving body is minimized, is contrasted to two sub-optimal systems. This is done by comparing fatigue life of a helical spring in the system for four cases. The optimization is based on minimizing vibrations, but it also increases the number of cycles to failure of the considered spring. A significant portion of this improvement is due to the inclusion of an optimally tuned inerter in the isolator.

**Keywords:** vibration isolation, fatigue life, inerter, finite element method, optimization

## 1. Introduction

Mechanical systems *e.g.* car suspension systems [1] are often subjected to high dynamic loading during their lifetime. Such service loadings can cause unwanted vibration and premature failure resulting from destructive fatigue mechanisms. These are especially evident in case of resonant harmonic excitations. Heavy-duty springs used in car suspension systems [1] are an example where a crack may initiate at a stress concentration location and propagate, leading to a potentially catastrophic fatigue failure, especially due to vibration-induced fatigue [2]-[7]. Beneficiary vibration-based optimizations with goal of targeted vibration reduction by using the minimization of kinetic energy were already performed and can be found in literature [8], [9]. Considering vibration fatigue, both stiffness and strength parameters of system should be determined, where broad literature considering helical springs can be found [10]-[20], also including exclusively numerical analysis - particularly finite element method (FEM) [21]. Both older seminal works on strength of materials [22], and modern mechanical engineering textbooks [23], [24] touch on the subject of spring fatigue. Considering springs as machine elements that need to withstand exceptionally long life, appropriate high-cycle fatigue (HCF) calculation method [25], [26] can be utilized for calculation [24], [27]. Broad literature on analyzing the fatigue life, particularly for helical springs can be observed [27]-[33]. Problem with unambiguously defining the stress field and corresponding fatigue life is that in literature, multiple versions of proposed stress correction factors for spring can be found [10]-[15], even by same authors. One of the most popular ones originate from Wahl [10], [11], [14], [15], [18]-[20], [22]-[24], Bergsträsser [14], [15], [18], [20], [23] and Göhner [11]-[13], [18], [20]. All three authors seem to be favorable choice in German literature [20]. It is interesting to note that German DIN standard used Göhner as a reference [12], [13], but then changed to Bergsträsser and Wahl [14], [15] in contemporary times. Discrepancy in using the appropriate correction factor can also be observed in dedicated spring fatigue literature, where Wahl himself in his seminal work [11] recommends that using his stress correction factor may result in overly-conservative fatigue prediction, while [19], [24] recommend using Wahl's stress correction factor especially for fatigue. Shigley [23] for instance recommends Bergsträsser for simplicity reasons. In the present study, investigations are conducted to model the fatigue load of a helical spring acting as an elastic element in a simple and physically transparent two degree-of-freedom (2-DOF) vibration isolation system. Both analytical/empirical and numerical methods are employed with help of specialized software packages like FEM based *Abaqus* [34] and *Fe-Safe* [35].

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## 2. Analytical and finite element vibration and fatigue analysis

The problem studied is represented by a lumped parameter model as shown in Fig. 1.a). It is assumed that the critical component concerning fatigue is a helical spring  $k_3$ , also shown in Fig. 1.b) where  $E$  is (Young) modulus of elasticity,  $\nu$  is Poisson's factor,  $S'_f$  is fatigue strength coefficient, while  $b$  is Basquin's exponent, *i.e.* fatigue strength exponent [25], [26]. Angle  $\alpha$  represents pitch angle and  $l$  is the pitch. Number of active coils is denoted as  $i$  ( $i = 2$  on Fig. 1.b) and  $h$  is spring length where  $h = li$ .  $D$  and  $d$  are large and small spring diameters respectively, and  $C = D/d$  is defined as spring index [11], [23], [24].  $D$  can also be designated as the mean coil diameter and  $d$  as the wire diameter [23]. Recommended values of spring index  $C$  for industrial purposes lie in between  $C = 4 \dots 12$  [23].

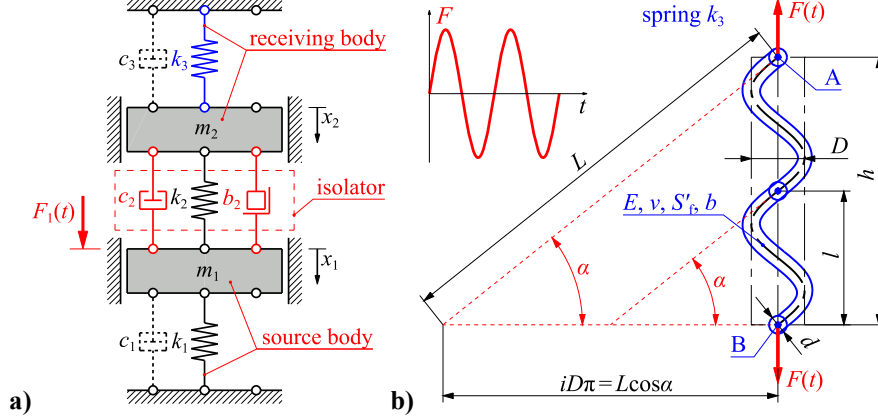


Fig. 1. a) The 2-DOF linear discrete vibration isolation system, b) helical spring  $k_3$

The goal of the vibration-based optimization is to minimize vibrations of the *receiving body* *i.e.* vibrations of mass  $m_2$  which come along with the deflections of spring  $k_3$  [8]. In this optimization, the excitation of the source body  $F(t)$  is assumed to have white noise spectral properties [8], *i.e.* unit loading amplitude  $F_0 = 1$  that can be scaled conveniently is adopted. The whole system consists of masses  $m_1$  and  $m_2$ , springs  $k_1$ ,  $k_2$  and  $k_3$ , a viscous damper  $c_2$  and an inerter of inertance  $b_2$ . The inerter produces a force  $F$  proportional to the relative acceleration between masses  $m_1$  and  $m_2$  [1]. Optimized parameters  $c_{2,opt}$ ,  $c_{2,opt2}$  and  $b_{2,opt}$  are obtained by minimizing the frequency averaged kinetic energy of the receiving body [8], [9]. Table 1. shows example parameters used in this isolator optimization process.

Table 1. The 2-DOF optimization model example parameters

$m_1$ , kg	$m_2$ , kg	$k_1$ , N/m	$k_2$ , N/m	$k_3$ , N/m	$c_{2,lo}$ , Ns/m	$c_{2,hi}$ , Ns/m
1	$m_1/2$	1	$k_1/10$	$k_1$	$c_{2,opt}/100$	$100c_{2,opt}$

Fig. 2. shows plotted numerical results of optimization process where minimum kinetic energy  $I_{k,min}$  is found for the cases without inerter (Fig. 2.a) -  $c_{2,opt}$ ) and with inerter (Fig. 2.b) -  $c_{2,opt2}$  and  $b_{2,opt}$ ). Obtained optimized parameters are further used in the fatigue calculation evaluation of spring  $k_3$ .

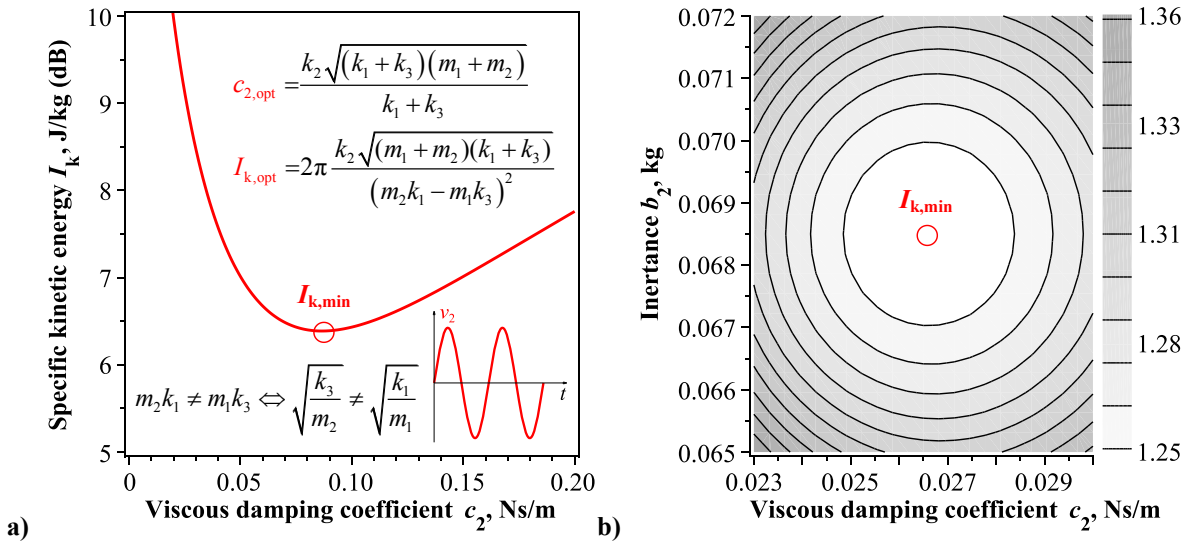
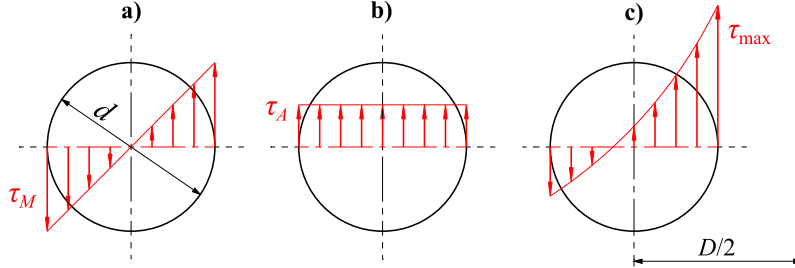


Fig. 2. Mass  $m_2$  specific kinetic energy index  $I_k$ : a)  $c_{2,opt}$ , b)  $c_{2,opt2}$  and  $b_{2,opt}$

Obtained velocity/displacement amplitudes in the frequency domain can now be tied to stress amplitudes necessary for performing fatigue analysis. Spring nominal displacement and stress [23] write as

$$k_{3,\text{nom}} = \frac{F}{\delta_{\text{nom}}} = \frac{Gd^4}{8D^3i} \Rightarrow \delta_{\text{nom}} = \frac{8FD^3i}{Gd^4}, \tau_{\text{nom}} = \frac{8FD}{\pi d^3}, \quad (1a,b)$$

where  $G = E/[2(1+\nu)]$  is the shear modulus. However, Eq. (1a,b) is obtained by viewing spring as a thin beam/rod loaded with torsion shear stress, where direct shear, curvature and pitch angle effects are ignored. Therefore, additional correction factors need to be applied for displacement and shear stress, where relations  $\delta_{\text{max}} = K_\delta \delta_{\text{nom}}$  and  $\tau_{\text{max}} = K_\tau \tau_{\text{nom}}$  hold. Fig. 3. Shows spring shear stress correction.



**Fig. 3. Spring shear stresses: a) torsion shear  $\tau_M$ , b) transverse/direct shear  $\tau_A$ , c) combined torsion and direct shear with curvature effect  $\tau_{\text{max}}$**

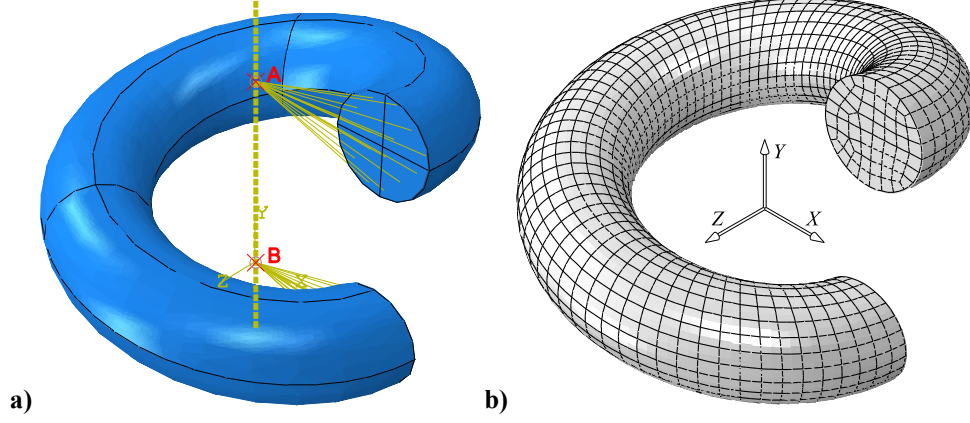
Authors give different correction factors depending on theory used [19]. Table 2. shows these factors.

**Table 2. Expressions for stress and deflection correction factors**

Author	Stress correction factor $K_\sigma$	Deflection correction factor $K_\delta$
<b>Strength of Materials</b>		
Wahl [11], DIN 13906 [14]	$\frac{4C-1}{4C-4} + \frac{0.615}{C}$	-
Honegger [18]	$\frac{C \cos(\alpha)}{C - \cos^2(\alpha)} + 0.615 \frac{\cos(\alpha)}{C}$	$\frac{2C^2 - \cos^4(\alpha)}{2C^2 \cos^5(\alpha)}$
<b>Elasticity Theory</b>		
Göhner [18], DIN 2089 [12]	$1 + \frac{5}{4C} + \frac{7}{8C^2} + \frac{1}{C^3}$	$\frac{\cos(\alpha)}{\left[1 + \frac{3}{16} \left(\frac{\cos^4(\alpha)}{C^2 - 1}\right)\right]^{-1}} + \frac{2G}{E} \sin(\alpha) \tan(\alpha)$
Ancker & Goodier [16]	$1 + \frac{5}{4C} + \frac{7}{8C^2} + \frac{1}{2} \tan^2(\alpha)$	$1 - \frac{3}{16C^2} + \frac{3+\nu}{2(1+\nu)} \tan^2(\alpha)$
<b>Approximate/empirical relation</b>		
Bergsträsser [18], DIN 13906 [14]	$\frac{C + 0.5 + \sin^2(\alpha)}{C - 0.75 + 1.51 \sin^2(\alpha)}$	-
<b>Strain Energy (Castigliano's) Method</b>		
Shigley [23]	-	$1 + \frac{1}{2C^2}$
Dym [17]	-	$\frac{1}{\cos(\alpha)} \left[ \left(1 + \frac{1}{2C^2}\right) \cos^2(\alpha) + \left(1 + \frac{1}{4C^2}\right) \frac{\sin^2(\alpha)}{(1+\nu)} \right]$

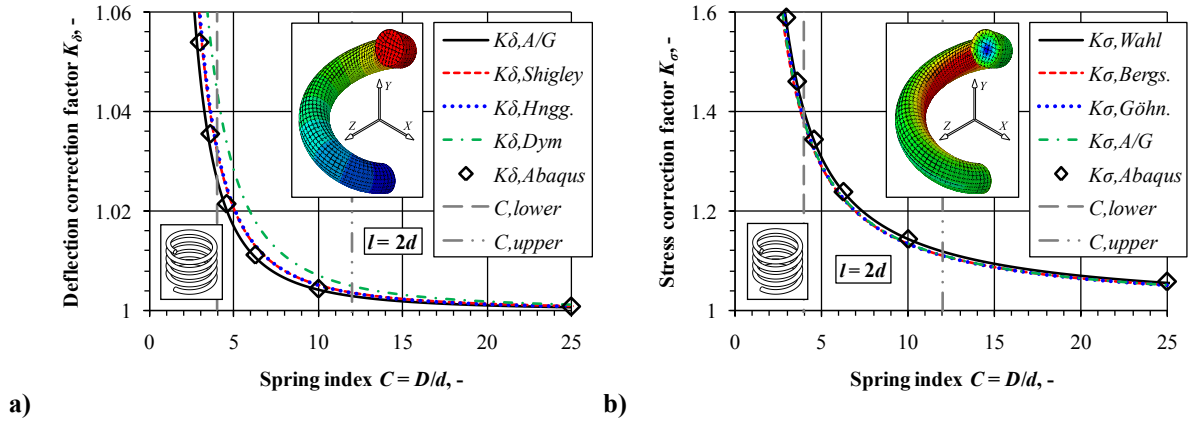
Since stresses in spring are mostly shear governed [23], [24], by adopting the von Mises criterion with relation  $\sigma_{\text{eqv(HMH),max}} = 3^{1/2} \tau_{\text{max}}$ , stress correction factor now reads as  $K_\sigma$ . In order to compare various deflection and stress correction factors from Table 2., referent relations are put to test using an alternative approach. All results are compared to those obtained using FEM software *Abaqus* [34]. *Abaqus* computational model consists of 3D second order 20 node hexahedron continuum elements C3D20R, which employ reduced integration and show superior performance due to additional nodes on sides of finite element [34]. Analysis is performed as linear and quasi-static, *i.e.* time is dimensionless. Con-

vergence study/mesh sensitivity check is performed beforehand and it is found that eight 2<sup>nd</sup> order hexahedron elements per spring thickness give highly accurate results for analyzed deflection/stress problem. Boundary conditions are defined through two reference points (RPs), analogue to Fig. 1.b), where pinned-pinned conditions are assumed. Full definition of boundary conditions is:  $B(u,v,w,\varphi_y=0)$ ,  $A(u,w=0)$ . Reference points A and B are coupled to belonging spring sides through kinematic attachment of type *distributing*, which allows for deformations of connection [34]. Fig. 4.a) shows computational model with fully defined reference points A and B, while Fig. 4.b) shows referent FE mesh.



**Fig. 4. Abaqus computational model,  $i = 1$ : a) RP(A,B) definition, b) C3D20R undeformed mesh**

For simplicity and computational effectiveness, only one coil is observed, *i.e.*  $i = 1$ . Six parametric *Abaqus* models are defined where spring pitch  $l = 2d$ . Fig. 5. shows all results in dimensionless form.



**Fig. 5. Spring correction factors parametric comparison: a) deflection factor  $K_\delta$ , b) stress factor  $K_\sigma$**

As shown in Fig. 5.a), Ancker & Goodier ( $K_{\delta,A/G}$ ) expression [16] shows best agreement with *Abaqus* model, while Shigley, Honegger and especially Dym somewhat *overestimate* the deflection. As shown in Fig. 5.b), Wahl ( $K_{\sigma,W}$ ) expression [10], [11] agrees excellently with *Abaqus* model, while Bergsträsser, Göhner and Ancker & Goodier somewhat *underestimate* the stress field, but show almost identical mutual results. Therefore, Ancker & Goodier model is adopted for deflection, and Wahl model is adopted for stress field. As already observed by Timoshenko [22], shear/equivalent stress is at maximum at inner side of spring coil, therefore potential crack initiation location is identified. One should also notice that stress field is homogenous through entire one coil spring, therefore terminology of stress *correction* is favorable compared to stress *concentration*. The fatigue analysis of the spring is considered next. Analytical and numerical stress results are used to calculate the number of cycles to failure according to the *von Mises* criterion [23], [24] in the context of *S-N i.e. Basquin's* curve [25], [26]. Type of spring processing and manufacture, *e.g.* favored shot-peening described by Shigley [23], Ugural [24] and Fatemi [25] among others, is not considered. Spring is for simplicity considered to be perfectly smooth and without any residual stresses. Table 3. shows example spring parameters.

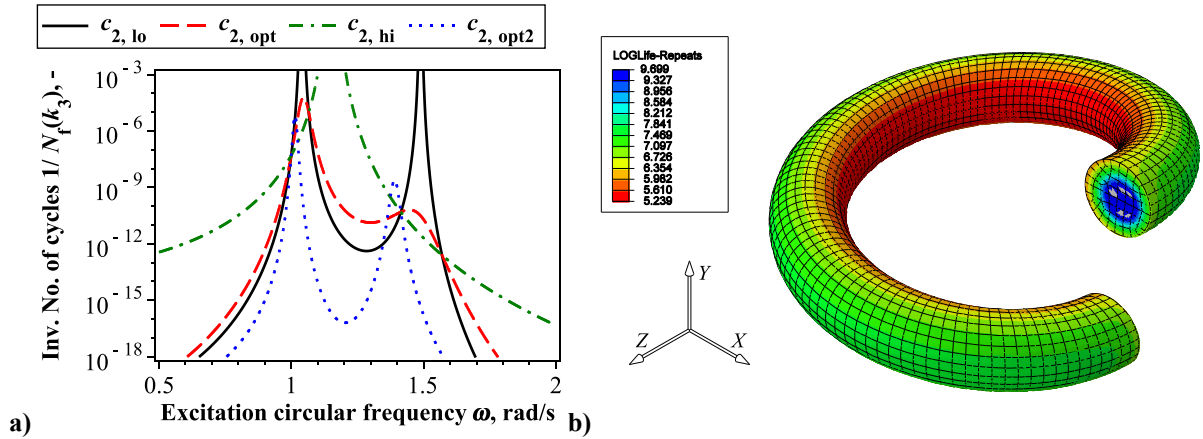
**Table 3. Helical spring computational model example parameters**

$D$ , mm	$d$ , mm	$l$ , mm	$i$ , -	$E$ , MPa	$\nu$ , -	$S'_B$ MPa	$b$ , -	$F_0$ , N
50	11	22	1	200 000	0	1 000	-0.1	1 344.352

Unconventional Poisson factor  $\nu = 0$  is adopted in Table 3. as stress field shouldn't depend on the used value, but robustness of adopted Ancker & Goodier displacement equation can be tested on an illustrative example.  $D$  and  $d$  are chosen so  $C = D/d \approx 4.545$  which is a relatively small spring index, but still in the industrially accepted boundaries. However, such small spring index  $C$  results with a relatively large stress correction factor which is a convenient fatigue benchmark. Rest of the material parameters ( $E$ ,  $S'_f$  and  $b$ ) are chosen in such way to represent regular fatigue and elastic properties of steel [25], [26]. Also, fatigue notch sensitivity is near unity, *i.e.*  $K_{t(\sigma)} = K_f$  which is a valid assumption according to Ugural [24]. Spring fatigue life can now be calculated according to derived simplified expression

$$N_f = \left( \frac{\sqrt{3}G}{\pi d C^2 i} \frac{K_{\sigma, W}}{K_{\delta, A/G}} \frac{\delta_{\max}}{S'_f} \right)^{\frac{1}{b}}, \quad (2)$$

where all correction factors are taken into account, and  $\delta_{\max}$  refers to displacement amplitude  $\delta_0(\omega)$ . For fatigue analysis, *Fe-Safe* [35] software suite is employed with *von Mises* criterion evoked, taking into account converged FEM nodal stresses from *Abaqus*. Custom material is defined according to Table 3. and entire spring is analyzed. Fig. 6.a) and b) shows final results of performed HCF analysis.



**Fig. 6. Spring  $k_3$ : a) inverse No. of cycles to fracture/failure  $1/N_f(k_3)$ , b) *Abaqus/Fe-Safe*  $N_f(k_3)$**

The improvement in the number of cycles to failure  $N_f$  is evident at most frequencies when using the optimum damping  $c_{2,opt}$  (Fig. 6.a), dashed line) in comparison to low damping  $c_{2,lo} = c_{2,opt}/100$  (solid line), or high damping  $c_{2,hi} = 100c_{2,opt}$  (dot-dashed line). Additionally, a significant further improvement in the fatigue life  $1/N_f(\omega)$  is observed at most frequencies, in case where the optimum inerter  $b_{2,opt}$  is implemented in combination with the optimum damper  $c_{2,opt2}$ , as shown in Fig. 6.a) (dotted line). Fig. 6.b) shows particular fatigue life results of spring with parameters obtained from Table 3. and post-processed in *Abaqus*. *Fe-Safe* shows homogenous life field scaled according to expression

$$(N_f)_{\text{LOGLife-Repeats}} = \log_{10}(N_f) \Leftrightarrow N_f = 10^{(N_f)_{\text{LOGLife-Repeats}}}, \quad (3)$$

where actual minimum number of cycles  $N_f$  compared to analytical results is shown in lower Table 4.

**Table 4. Analytical and numerical results fatigue comparison**

Method	$\sigma_{\text{eqv}}$ , MPa	$E_{\text{rel}}$ , %	$\delta_{\max}$ , mm	$E_{\text{rel}}$ , %	$N_f$ -	$E_{\text{rel}}$ , %
Analytical	300.000	-	0.936895	-	169350.9	-
<i>Abaqus</i>	299.307	0.232	0.937889	-0.106	173315.8	0.001654
<i>Fe-Safe</i>	-	-	-	-	173318.7	-2.28932

It can be seen that force amplitude  $F_0$  from Table 3. is chosen in such way to result in max. equivalent stress  $\sigma_{\text{eqv}(\max)} = 300$  MPa in Table 4., according to adopted Wahl criterion. Although analytical and numerical stress  $\sigma_{\text{eqv}(\max)}$  results shown on Table 4. differ by only  $\sim 0.232$  %, that difference is enlarged in fatigue analysis to  $\sim 2.289$  %, as fatigue expression (2) presents exponential equation where small differences in stress result in much larger dissipation in fatigue analysis. By taking *Abaqus* numerical results into account, difference between hand calculated fatigue and *Fe-Safe* results fall to  $\sim 0.0016$  % which is negligible and can be attributed to rounding error. It can be seen that Wahl prediction gives somewhat more conservative results compared to numerical results. Finally, Ancker & Goodier deflec-



tion correction excellently predicts displacement field where maximum relative error is around 0.1 %. However, the present optimization is based on a vibration-based criterion. It would be interesting as a future work, to consider a type of optimization which would aim at maximizing fatigue life of the spring and compare it to the present results. Also, considering that pitch angle is defined through relation  $l = 2d$  and  $\alpha = \tan^{-1} [l/(\pi D)]$  in this work, it would be beneficiary to investigate the influence of pitch angle on deflection and stress correction, as some of the expressions from Table 2. consider the pitch angle influence, and some don't. That most notably applies to further testing of Wahl stress correction factor, as it currently demonstrates highest accuracy for chosen parameters. Logical continuation of this work would also be investigation of mean stress influence on the spring fatigue life, as all calculations were performed for simple harmonic *i.e.* fully reversed loading  $R = -1$  where dead weight static load wasn't considered.

### 3. Conclusions

If the parameters of the passive, inerter based isolator are optimized to maximize the effect of vibration isolation, this also seems to correspond to significant reductions of the stresses in the considered spring and an increase of its fatigue life as a consequence. The deflections and stresses in the spring have been calculated numerically and analytically, and they agree very well. Therefore it can be concluded that minimizing the kinetic energy of the receiving body is beneficial in terms of the spring fatigue life. However, it would be most interesting to investigate whether the vibration-based optimization can be met in parallel with a fatigue-life-based optimization, or if there would be some trade-off regarding the achievement of the two goals simultaneously.

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### References

- [1] Smith, M. C., Synthesis of mechanical networks: The inerter, IEEE Trans. on Automatic Control 47(10) (2002) 1648–1662.
- [2] Steinberg, D. S., Vibration Analysis for Electronic Equipment, Third Edition, John Wiley & Sons, New York, 2000.
- [3] Bishop, N. W. M., Sherratt, F., Finite Element Based Fatigue Calculations, NAFEMS Ltd, Farnham, 2000.
- [4] Lalanne, C., Fatigue Damage, Mechanical Vibration and Shock Analysis, Volume 4, Second Edition, Wiley-ISTE, London, 2009.
- [5] Thorby, D., Structural Dynamics and Vibration in Practice, An Engineering Handbook, Butterworth-Heinemann, New York, 2008.
- [6] Lee, Y., Barkey, M. E., Kang, H., Metal Fatigue Analysis Handbook, Practical Problem-Solving Techniques for Computer-Aided Engineering, Elsevier Inc., Waltham, 2012.
- [7] Mršnik, M., Slavič, J., Boltežar, M., Frequency-Domain Methods for a Vibration-Fatigue-Life Estimation - Application to Real Data, International Journal of Fatigue 47 (2013) 8–17.
- [8] Alujević, N., Wolf, H., Gardonio, P., Tomac, I., Stability and performance limits for active vibration isolation using blended velocity feedback, JSV 330(21) (2011) 4981–4997.
- [9] Alujević, N., Zhao, G., Depraetere, B., Sas, P., Pluymers, B., Desmet, W., H2 optimal vibration control using inertial actuators and a comparison with tuned mass dampers, JSV 333 (2014) 4073–4083.
- [10] Wahl, A. M., Helical Compression and Tension Springs, ASME Paper A-38, J. Appl. Mech. 2(1) (1935) A-35-A-37.
- [11] Wahl, A. M., Mechanical Springs, First Edition, Penton Pub. Co., Cleveland, 1944.
- [12] DIN EN 2089-1-1963-01, Helical springs made from round wire and rod - Calculation and design of Compression springs, Februar 1963.
- [13] DIN EN 2089-1-1963-02, Helical springs made from round wire and rod - Calculation and design of Tension springs, Februar 1963.
- [14] DIN EN 13906-1, Cylindrical helical springs made from round wire and bar - Calculation and design - Part 1: Compression springs, Beuth Verlag, Berlin, 2002.

- [15] DIN EN 13906-2, Cylindrical helical springs made from round wire and bar - Calculation and design - Part 2: Extension springs, Beuth Verlag, Berlin, 2002.
- [16] Ancker, C. J. Jr., Goodier, J. N., Pitch and Curvature Correction for Helical Springs, *ASME J. Appl. Mech.* 25(4) (1958) 466–470.
- [17] Dym, C. L., Consistent derivations of spring rates for helical springs, *ASME Journal of Mechanical Design* 131(7) (2009) 1–5.
- [18] Research Committee on the Analysis of Helical Spring, Report of Research Committee on the Analysis of Helical Spring, *Transactions of Japan Society of Spring Engineers* 2004(49) (2004) 35–75.
- [19] Society of Automotive Engineers (SAE), Spring Design Manual, Ae Series, Society of Automotive Engineers Inc., Warrendale, 1990.
- [20] Meissner, M., Schorcht, H.-J., Kletzlin, U., *Metallfedern: Grundlagen, Werkstoffe, Berechnung, Gestaltung und Rechneinsatz*, 3. Auflage, Springer-Verlag, Berlin, Heidelberg, 2015.
- [21] Shimoseki, M., Hamano, T., Imaizumi, T., *FEM for Springs*, Springer-Verlag, Berlin, 2003.
- [22] Timoshenko, S. P., *Strength Of Materials, Part I, Elementary Theory and Problems*, Second Edition, D. Van Nostrand. Company, Inc., New York, 1940.
- [23] Budynas, R. G., Nisbett, J. K., *Shigley's Mechanical Engineering Design*, 10th Edition, McGraw-Hill, New York, 2015.
- [24] Ugural, A. C., *Mechanical Design of Machine Components*, Second Edition, CRC Press, Boca Raton, 2015.
- [25] Stephens, R. I., Fatemi, A., Stephens, R. R., Fuchs, H. O., *Metal Fatigue in Engineering*, Second Edition, John Wiley & Sons, Inc., New York, 2005.
- [26] Roessle, M. L., Fatemi, A., Strain-controlled fatigue properties of steels and some simple approximations, *International Journal of Fatigue* 22(6) (2000) 495–511.
- [27] Berger C., Kaiser B., Result of very high cycle fatigue tests on helical compression springs, *International Journal of Fatigue* 28 (2006) 1658–1663.
- [28] Kaiser, B., Berger, C., Fatigue behaviour of technical springs, *Materialwissenschaft und Werkstofftechnik* 36(11) (2005) 685–696.
- [29] Del Llano-Vizcaya, L., Rubio-González, C., Mesmacque, G., Cervantes-Hernandez, T., Multiaxial fatigue and failure analysis of helical compression springs, *Engineering Failure Analysis* 13(8) (2006) 1303–1313.
- [30] Pyttel, B., Ray, K. K., Brunner, I., Tiwari, A., Kaoua, S. A., Investigation of probable failure position in helical compression springs used in fuel injection system of diesel engines, *IOSR J. Mech. Civil Eng.* 2(3) (2012) 24–29.
- [31] Rivera, R., Chiminelli, A., Gómez, C., Núñez, J. L., Fatigue failure analysis of a spring for elevator doors, *Engineering Failure Analysis* 17(4) (2010) 731–738.
- [32] Ružička, M., Doubrava, K., Loading regimes and designing helical coiled springs for safe fatigue life, *Res. Agr. Eng.* 51(2) (2005) 50–55.
- [33] Kamal, M., Rahman, M. M., Finite element-based fatigue behaviour of springs in automobile suspension, *International Journal of Automotive and Mechanical Engineering (IJAME)* 10 (2014) 1910-1919.
- [34] Dassault Systèmes, *Abaqus 6.9 User's guide and theoretical manual*, Hibbitt, Karlsson & Sorensen, Inc., 2009.
- [35] SafeTechnology, *Fe-Safe 6 User Manual*, 2011.